

# Varieties of groupoids and quasigroups generated by linear-bivariate polynomials over the ring $\mathbb{Z}_n^{*\dagger}$

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## Abstract

Some varieties of groupoids and quasigroups generated by linear-bivariate polynomials  $P(x, y) = a + bx + cy$  over the ring  $\mathbb{Z}_n$  are studied. Necessary and sufficient conditions for such groupoids and quasigroups to obey identities which involve one, two, three (e.g. Bol-Moufang type) and four variables w.r.t.  $a$ ,  $b$  and  $c$  are established. Necessary and sufficient conditions for such groupoids and quasigroups to obey some inverse properties w.r.t.  $a$ ,  $b$  and  $c$  are also established. This class of groupoids and quasigroups are found to belong to some varieties of groupoids and quasigroups such as medial groupoid(quasigroup), F-quasigroup, semi automorphic inverse property groupoid(quasigroup) and automorphic inverse property groupoid(quasigroup).

## 1 Introduction

### 1.1 Groupoids, Quasigroups and Identities

Let  $G$  be a non-empty set. Define a binary operation  $(\cdot)$  on  $G$ .  $(G, \cdot)$  is called a groupoid if  $G$  is closed under the binary operation  $(\cdot)$ . A groupoid  $(G, \cdot)$  is called a quasigroup if the

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equations  $a \cdot x = b$  and  $y \cdot c = d$  have unique solutions for  $x$  and  $y$  for all  $a, b, c, d \in G$ . A quasigroup  $(G, \cdot)$  is called a loop if there exists a unique element  $e \in G$  called the identity element such that  $x \cdot e = e \cdot x = x$  for all  $x \in G$ .

A function  $f : S \times S \rightarrow S$  on a finite set  $S$  of size  $n > 0$  is said to be a Latin square (of order  $n$ ) if for any value  $a \in S$  both functions  $f(a, \cdot)$  and  $f(\cdot, a)$  are permutations of  $S$ . That is, a Latin square is a square matrix with  $n^2$  entries of  $n$  different elements, none of them occurring more than once within any row or column of the matrix.

**Definition 1.1** *A pair of Latin squares  $f_1(\cdot, \cdot)$  and  $f_2(\cdot, \cdot)$  is said to be orthogonal if the pairs  $(f_1(x, y), f_2(x, y))$  are all distinct, as  $x$  and  $y$  vary.*

For associative binary systems, the concept of an inverse element is only meaningful if the system has an identity element. For example, in a group  $(G, \cdot)$  with identity element  $e \in G$ , if  $x \in G$  then the inverse element for  $x$  is the element  $x^{-1} \in G$  such that

$$x \cdot x^{-1} = x^{-1} \cdot x = e.$$

In a loop  $(G, \cdot)$  with identity element  $e$ , the left inverse element of  $x \in G$  is the element  $x^\lambda \in G$  such that

$$x^\lambda \cdot x = e$$

while the right inverse element of  $x \in G$  is the element  $x^\rho \in G$  such that

$$x \cdot x^\rho = e$$

In case  $(G, \cdot)$  is a quasigroup, then for each  $x \in G$ , the elements  $x^\rho \in G$  and  $x^\lambda \in G$  such that  $xx^\rho = e^\rho$  and  $x^\lambda x = e^\lambda$  are called the right and left inverse elements of  $x$  respectively. Here,  $e^\rho \in G$  and  $e^\lambda \in G$  satisfy the relations  $xe^\rho = x$  and  $e^\lambda x = x$  for all  $x \in G$  and are respectively called the right and left identity elements. Whenever  $e^\rho = e^\lambda$ , then  $(G, \cdot)$  becomes a loop.

In case  $(G, \cdot)$  is a groupoid, then for each  $x \in G$ , the elements  $x^\rho \in G$  and  $x^\lambda \in G$  such that  $xx^\rho = e_\rho(x)$  and  $x^\lambda x = e_\lambda(x)$  are called the right and left inverse elements of  $x$  respectively. Here,  $e_\rho(x) \in G$  and  $e_\lambda(x) \in G$  satisfy the relations  $xe_\rho(x) = x$  and  $e_\lambda(x)x = x$  for each  $x \in G$  and are respectively called the local right and local left identity elements of  $x$ . Whenever  $e_\rho(x) = e_\lambda(x)$ , then we simply write  $e(x) = e_\rho(x) = e_\lambda(x)$  and call it the local identity of  $x$ .

The basic text books on quasigroups, loops are Pflugfelder [19], Bruck [1], Chein, Pflugfelder and Smith [2], Dene and Keedwell [3], Goodaire, Jespers and Milies [4], Sabinin [25], Smith [26], Jaíy  l   [5] and Vasantha Kandasamy [28].

Groupoids, quasigroups and loops are usually studied relative to properties or identities. If a groupoid, quasigroup or loop obeys a particular identity, then such types of groupoids, quasigroups or loops are said to form a variety. In this work, our focus will be on groupoids and quasigroups. Some identities that describe groupoids and quasigroups which would be of interest to us here are categorized as follows:

(A) Those identities which involve one element only on each side of the equality sign:

$$aa = a \quad \text{idempotent law} \quad (1)$$

$$aa = bb \quad \text{unipotent law} \quad (2)$$

(B) Those identities which involve two elements on one or both sides of the equality sign:

$$ab = ba \quad \text{commutative law} \quad (3)$$

$$(ab)b = a \quad \text{Sade right Keys law} \quad (4)$$

$$b(ba) = a \quad \text{Sade left keys law} \quad (5)$$

$$(ab)b = a(bb) \quad \text{right alternative law} \quad (6)$$

$$b(ba) = (bb)a \quad \text{left alternative law} \quad (7)$$

$$a(ba) = (ab)a \quad \text{medial alternative law} \quad (8)$$

$$a(ba) = b \quad \text{law of right semisymmetry} \quad (9)$$

$$(ab)a = b \quad \text{law of left semisymmetry} \quad (10)$$

$$a(ab) = ba \quad \text{Stein first law} \quad (11)$$

$$a(ba) = (ba)a \quad \text{Stein second law} \quad (12)$$

$$a(ab) = (ab)b \quad \text{Schroder first law} \quad (13)$$

$$(ab)(ba) = a \quad \text{Schroder second law} \quad (14)$$

$$(ab)(ba) = b \quad \text{Stein third law} \quad (15)$$

$$ab = a \quad \text{Sade right translation law} \quad (16)$$

$$ab = b \quad \text{Sade left translation law} \quad (17)$$

(C) Those identities which involve three distinct elements on one or both sides of the equality sign:

$$(ab)c = a(bc) \quad \text{associative law} \quad (18)$$

$$a(bc) = c(ab) \quad \text{law of cyclic associativity} \quad (19)$$

$$(ab)c = (ac)b \quad \text{law of right permutability} \quad (20)$$

$$a(bc) = b(ac) \quad \text{law of left permutability} \quad (21)$$

$$a(bc) = c(ba) \quad \text{Abel-Grassman law} \quad (22)$$

$$(ab)c = a(cb) \quad \text{commuting product law} \quad (23)$$

$$c(ba) = (bc)a \quad \text{dual of commuting product} \quad (24)$$

$$(ab)(bc) = ac \quad \text{Stein fourth law} \quad (25)$$

$$(ba)(ca) = bc \quad \text{law of right transitivity} \quad (26)$$

$$\begin{aligned}
(ab)(ac) &= bc && \text{law of left transitivity} && (27) \\
(ab)(ac) &= cb && \text{Schweitzer law} && (28) \\
(ba)(ca) &= cb && \text{dual of Schweitzer law} && (29) \\
(ab)c &= (ac)(bc) && \text{law of right self-distributivity law} && (30) \\
c(ba) &= (cb)(ca) && \text{law of left self-distributivity law} && (31) \\
(ab)c &= (ca)(bc) && \text{law of right abelian distributivity} && (32) \\
c(ba) &= (cb)(ac) && \text{law of left abelian distributivity} && (33) \\
(ab)(ca) &= [a(bc)]a && \text{Bruck-Moufang identity} && (34) \\
(ab)(ca) &= a[bc]a && \text{dual of Bruck-Moufang identity} && (35) \\
[(ab)c]b &= a[b(cb)] && \text{Moufang identity} && (36) \\
[(bc)b]a &= b[c(ba)] && \text{Moufang identity} && (37) \\
[(ab)c]b &= a[(bc)b] && \text{right Bol identity} && (38) \\
[b(cb)]a &= b[c(ba)] && \text{left Bol identity} && (39) \\
[(ab)c]a &= a[b(ca)] && \text{extra law} && (40) \\
[(ba)a]c &= b[(aa)c] && \text{RC}_4 \text{ law} && (41) \\
[b(aa)]c &= b[a(ac)] && \text{LC}_4 \text{ law} && (42) \\
(aa)(bc) &= [a(ab)]c && \text{LC}_2 \text{ law} && (43) \\
[(bc)a]a &= b[(ca)a] && \text{RC}_1 \text{ law} && (44) \\
[a(ab)]c &= a[a(bc)] && \text{LC}_1 \text{ law} && (45) \\
(bc)(aa) &= b[(ca)a] && \text{RC}_2 \text{ law} && (46) \\
[(aa)b]c &= a[a(bc)] && \text{LC}_3 \text{ law} && (47) \\
[(bc)a]a &= b[c(aa)] && \text{RC}_3 \text{ law} && (48) \\
[(ba)a]c &= b[a(ac)] && \text{C-law} && (49) \\
a[b(ca)] &= cb && \text{Tarski law} && (50) \\
a[(bc)(ba)] &= c && \text{Neumann law} && (51) \\
(ab)(ca) &= (ac)(ba) && \text{specialized medial law} && (52)
\end{aligned}$$

(D) Those involving four elements:

$$(ab)(cd) = (ad)(cb) \quad \text{first rectangle rule} \quad (53)$$

$$(ab)(ac) = (db)(dc) \quad \text{second rectangle rule} \quad (54)$$

$$(ab)(cd) = (ac)(bd) \quad \text{internal mediality or medial law} \quad (55)$$

(E) Those involving left or right inverse elements:

$$x^\lambda \cdot xy = y \quad \text{left inverse property} \quad (56)$$

$$yx \cdot x^\rho = y \quad \text{right inverse property} \quad (57)$$

$$x(yx)^\rho = y^\rho \text{ or } (xy)^\lambda x = y^\lambda \quad \text{weak inverse property(WIP)} \quad (58)$$

$$xy \cdot x^\rho = y \text{ or } x \cdot yx^\rho = y \text{ or } x^\lambda \cdot (yx) = y \text{ or } x^\lambda y \cdot x = y \text{ cross inverse property(CIP)} \quad (59)$$

$$(xy)^\rho = x^\rho y^\rho \text{ or } (xy)^\lambda = x^\lambda y^\lambda \text{ automorphic inverse property (AIP)} \quad (60)$$

$$(xy)^\rho = y^\rho x^\rho \text{ or } (xy)^\lambda = y^\lambda x^\lambda \text{ anti-automorphic inverse property (AAIP)} \quad (61)$$

$$(xy \cdot x)^\rho = x^\rho y^\rho \cdot x^\rho \text{ or } (xy \cdot x)^\lambda = x^\lambda y^\lambda \cdot x^\lambda \text{ semi-automorphic inverse property (SAIP)} \quad (62)$$

**Definition 1.2** (*Trimedial Quasigroup*)

A quasigroup is trimedial if every subquasigroup generated by three elements is medial.

Medial quasigroups have also been called abelian, entropic, and other names, while tri-medial quasigroups have also been called triabelian, terentropic, etc.

There are two distinct, but related, generalizations of trimedial quasigroups. The variety of semimedial quasigroups(also known as weakly abelian, weakly medial, etc.) is defined by the equations

$$xx \cdot yz = xy \cdot xz \quad (63)$$

$$zy \cdot xx = zx \cdot yz \quad (64)$$

**Definition 1.3** (*Semimedial Quasigroup*)

A quasigroup satisfying (63) (resp. (64)) is said to be left (resp. right) semimedial.

**Definition 1.4** (*Medial-Like Identities*)

A groupoid or quasigroup is called an external medial groupoid or quasigroup if it obeys the identity

$$ab \cdot cd = db \cdot ca \quad \text{external medial or paramediality law} \quad (65)$$

A groupoid or quasigroup is called a palindromic groupoid or quasigroup if it obeys the identity

$$ab \cdot cd = dc \cdot ba \quad \text{palidromity law} \quad (66)$$

Other medial like identities of the form  $(ab)(cd) = (\pi(a)\pi(b))(\pi(c)\pi(d))$ , where  $\pi$  is a certain permutation on  $\{a, b, c, d\}$  are given as follows:

$$ab \cdot cd = ab \cdot dc \quad C_1 \quad (67)$$

$$ab \cdot cd = ba \cdot cd \quad C_2 \quad (68)$$

$$ab \cdot cd = ba \cdot dc \quad C_3 \quad (69)$$

$$ab \cdot cd = cd \cdot ab \quad C_4 \quad (70)$$

$$ab \cdot cd = cd \cdot ba \quad C_5 \quad (71)$$

$$ab \cdot cd = dc \cdot ab \quad C_6 \quad (72)$$

$$ab \cdot cd = ac \cdot db \quad CM_1 \quad (73)$$

$$ab \cdot cd = ad \cdot bc \quad CM_2 \quad (74)$$

$$ab \cdot cd = ad \cdot cb \quad CM_3 \quad (75)$$

$$ab \cdot cd = bc \cdot ad \quad CM_4 \quad (76)$$

$$ab \cdot cd = bc \cdot da \quad CM_5 \quad (77)$$

$$ab \cdot cd = bd \cdot ac \quad CM_6 \quad (78)$$

$$ab \cdot cd = bd \cdot ca \quad CM_7 \quad (79)$$

$$ab \cdot cd = ca \cdot bd \quad CM_8 \quad (80)$$

$$ab \cdot cd = ca \cdot db \quad CM_9 \quad (81)$$

$$ab \cdot cd = cb \cdot ad \quad CM_{10} \quad (82)$$

$$ab \cdot cd = cb \cdot da \quad CM_{11} \quad (83)$$

$$ab \cdot cd = da \cdot bc \quad CM_{12} \quad (84)$$

$$ab \cdot cd = da \cdot cb \quad CM_{13} \quad (85)$$

$$ab \cdot cd = db \cdot ac \quad CM_{14} \quad (86)$$

The variety of F-quasigroups was introduced by Murdoch [18].

**Definition 1.5** (*F-quasigroup*)

An F-quasigroup is a quasigroup that obeys the identities

$$x \cdot yz = xy \cdot (x \setminus x)z \quad \text{left } F\text{-law} \quad (87)$$

$$zy \cdot x = z(x/x) \cdot yx \quad \text{right } F\text{-law} \quad (88)$$

A quasigroup satisfying (87) (resp. (88)) is called a left (resp. right) F-quasigroup.

**Definition 1.6** (*E-quasigroup*)

An E-quasigroup is a quasigroup that obeys the identities

$$x \cdot yz = e_\lambda(x)y \cdot xz \quad E_l \text{ law} \quad (89)$$

$$zy \cdot x = zx \cdot ye_\rho(x) \quad E_r \text{ Law} \quad (90)$$

A quasigroup satisfying (89) (resp. (90)) is called a left (resp. right) E-quasigroup.

Some identities will make a quasigroup to be a loop, such are discussed in Keedwell [6, 7].

**Definition 1.7** (*Linear Quasigroup and T-quasigroup*) A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = x\alpha + y\beta + c$  where  $(Q, +)$  is a group,  $\alpha$  is its automorphism and  $\beta$  is a permutation of the set  $Q$ , is called a left linear quasigroup.

A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = x\alpha + y\beta + c$  where  $(Q, +)$  is a group,  $\beta$  is its automorphism and  $\alpha$  is a permutation of the set  $Q$ , is called a right linear quasigroup.

A T-quasigroup is a quasigroup  $(Q, \cdot)$  defined over an abelian group  $(Q, +)$  by  $x \cdot y = c + x\alpha + y\beta$ , where  $c$  is a fixed element of  $Q$  and  $\alpha$  and  $\beta$  are both automorphisms of the group  $(Q, +)$ .

Whenever one considers mathematical objects defined in some abstract manner, it is usually desirable to determine that such objects exist. Although occasionally this is accomplished by means of an abstract existential argument, most frequently, it is carried out through the presentation of a suitable example, often one which has been specifically constructed for the purpose. An example is the solution to the open problem of the axiomization of rectangular quasigroups and loops by Kinyon and Phillips [12] and the axiomization of trimedial quasigroups by Kinyon and Phillips [10, 11].

Chein et. al. [2] presents a survey of various methods of construction which has been used in the literature to generate examples of groupoids and quasigroups. Many of these constructions are ad hoc-designed specifically to produce a particular example; while others are of more general applicability. More can be found on the construction of  $(r, s, t)$ -inverse quasigroups in Keedwell and Shcherbacov [8, 9], idempotent medial quasigroups in Krčadinac and Volenec [14] and quasigroups of Bol-Moufang type in Kunen [15, 16].

**Remark 1.1** In the survey of methods of construction of varieties and types of quasigroups highlighted in Chein et. al. [2], it will be observed that some other important types of quasigroups that obey identities (1) to (90) are not mentioned. Also, examples of methods of construction of such varieties that are groupoids are also scarce or probably not in existence by our search. In Theorem 1.4 of Kirnasovsky [13], the author characterized T-quasigroups with a score and two identities from among identities (1) to (90). The present work thus proves some results with which such groupoids and quasigroups can be constructed.

## 1.2 Univariate and Bivariate Polynomials

Consider the following definitions.

**Definition 1.8** A polynomial  $P(x) = a_0 + a_1x + \cdots + a_nx^n$ ,  $n \in \mathbb{N}$  is said to be a permutation polynomial over a finite ring  $R$  if the mapping defined by  $P$  is a bijection on  $R$ .

**Definition 1.9** A bivariate polynomial is a polynomial in two variables,  $x$  and  $y$  of the form  $P(x, y) = \sum_{i,j} a_{ij}x^i y^j$ .

**Definition 1.10** (*Bivariate Polynomial Representing a Latin Square*)

A bivariate polynomial  $P(x, y)$  over  $\mathbb{Z}_n$  is said to represent (or generate) a Latin square if  $(\mathbb{Z}_n, *)$  is a quasigroup where  $*$  :  $\mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  is defined by  $x * y = P(x, y)$  for all  $x, y \in \mathbb{Z}_n$ .

Mollin and Small [17] considered the problem of characterizing permutation polynomials. They established conditions on the coefficients of a polynomial which are necessary and sufficient for it to represent a permutation.

Shortly after, Rudolf and Mullen [23] provided a brief survey of the main known classes of permutation polynomials over a finite field and discussed some problems concerning permutation polynomials (PPs). They described several applications of permutations which indicated why the study of permutations is of interest. Permutations of finite fields have become of considerable interest in the construction of cryptographic systems for the secure transmission of data. Thereafter, the same authors in their paper [24], described some results that had appeared after their earlier work including two major breakthroughs.

Rivest [22] studied permutation polynomials over the ring  $(\mathbb{Z}_n, +, \cdot)$  where  $n$  is a power of 2:  $n = 2^w$ . This is based on the fact that modern computers perform computations modulo  $2^w$  efficiently (where  $w = 2, 8, 16, 32$  or  $64$  is the word size of the machine), and so it was of interest to study PPs modulo a power of 2. Below is an important result from his work which is relevant to the present study.

**Theorem 1.1** (*Rivest [22]*)

*A bivariate polynomial  $P(x, y) = \sum_{i,j} a_{ij} x^i y^j$  represents a Latin square modulo  $n = 2^w$ , where  $w \geq 2$ , if and only if the four univariate polynomials  $P(x, 0)$ ,  $P(x, 1)$ ,  $P(0, y)$ , and  $P(1, y)$  are all permutation polynomial modulo  $n$ .*

Vadiraaja and Shankar [27] motivated by the work of Rivest continued the study of permutation polynomials over the ring  $(\mathbb{Z}_n, +, \cdot)$  by studying Latin squares represented by linear and quadratic bivariate polynomials over  $\mathbb{Z}_n$  when  $n \neq 2^w$  with the characterization of some PPs. Some of the main results they got are stated below.

**Theorem 1.2** (*Vadiraaja and Shankar [27]*)

*A bivariate linear polynomial  $a + bx + cy$  represents a Latin square over  $\mathbb{Z}_n$ ,  $n \neq 2^w$  if and only if one of the following equivalent conditions is satisfied:*

- (i) *both  $b$  and  $c$  are coprime with  $n$ ;*
- (ii)  *$a + bx$ ,  $a + cy$ ,  $(a + c) + bx$  and  $(a + b) + cy$  are all permutation polynomials modulo  $n$ .*

**Remark 1.2** *It must be noted that  $P(x, y) = a + bx + cy$  represents a groupoid over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a quasigroup over  $\mathbb{Z}_n$  if and only if  $(\mathbb{Z}_n, P)$  is a  $T$ -quasigroup. Hence whenever  $(\mathbb{Z}_n, P)$  is a groupoid and not a quasigroup,  $(\mathbb{Z}_n, P)$  is neither a  $T$ -quasigroup nor left linear quasigroup nor right linear quasigroup. Thus, the present study considers both  $T$ -quasigroup and non- $T$ -quasigroup.*

**Theorem 1.3** (*Vadiraaja and Shankar [27]*)

*If  $P(x, y)$  is a bivariate polynomial having no cross term, then  $P(x, y)$  gives a Latin square if and only if  $P(x, 0)$  and  $P(0, y)$  are permutation polynomials.*



The authors were able to establish the fact that Rivest's result for a bivariate polynomial over  $\mathbb{Z}_n$  when  $n = 2^w$  is true for a linear-bivariate polynomial over  $\mathbb{Z}_n$  when  $n \neq 2^w$ . Although the result of Rivest was found not to be true for quadratic-bivariate polynomials over  $\mathbb{Z}_n$  when  $n \neq 2^w$  with the help of counter examples, nevertheless some of such squares can be forced to be Latin squares by deleting some equal numbers of rows and columns.

Furthermore, Vadiraja and Shankar [27] were able to find examples of pairs of orthogonal Latin squares generated by bivariate polynomials over  $\mathbb{Z}_n$  when  $n \neq 2^w$  which was found impossible by Rivest for bivariate polynomials over  $\mathbb{Z}_n$  when  $n = 2^w$ .

### 1.3 Some Important Results on Medial-Like Identities

Some important results which we would find useful in our study are stated below.

**Theorem 1.4** (Polonijo [21])

*For any groupoid  $(Q, \cdot)$ , any two of the three identities (55), (65) and (66) imply the third one.*

**Theorem 1.5** (Polonijo [21])

*Let  $(Q, \cdot)$  be a commutative groupoid. Then  $(Q, \cdot)$  is palindromic. Furthermore, the constraints (55) and (65) are equivalent, i.e a commutative groupoid  $(Q, \cdot)$  is internally medial if and only if it is externally medial.*

**Theorem 1.6** (Polonijo [21])

*For any quasigroup  $(Q, \cdot)$  and  $i \in \{1, 2, \dots, 6\}$ ,  $C_i$  is valid if and only if the quasigroup is commutative.*

**Theorem 1.7** (Polonijo [21])

*For any quasigroup  $(Q, \cdot)$  and  $i \in \{1, 2, \dots, 14\}$ ,  $CM_i$  holds if and only if the quasigroup is both commutative and internally medial.*

**Theorem 1.8** (Polonijo [21])

*For any quasigroup  $(Q, \cdot)$  and  $i \in \{1, 2, \dots, 14\}$ ,  $CM_i$  is valid if and only if the quasigroup is both commutative and externally medial.*

**Theorem 1.9** *A quasigroup  $(Q, \cdot)$  is palindromic if and only if there exists an automorphism  $\alpha$  such that*

$$\alpha(x \cdot y) = y \cdot x \quad \forall x, y \in Q$$

*holds.*

It is important to study the characterization of varieties of groupoids and quasigroups represented by linear-bivariate polynomials over the ring  $\mathbb{Z}_n$  even though very few of such have been sighted as examples in the past.

## 2 Main Results

**Theorem 2.1** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\{\mathbb{Z}_n, \mathbb{Z}_p\}$  such that "HYPO" is true.  $P(x, y)$  represents a "NAME" {groupoid, quasigroup}  $\{(\mathbb{Z}_n, P), (\mathbb{Z}_p, P)\}$  over  $\{\mathbb{Z}_n, \mathbb{Z}_p\}$  if and only if "N and S" is true. (Table 1)*

### Proof

There are 66 identities for which the theorem above is true for in a groupoid or quasigroup. For the sake of space, we shall only demonstrate the proof for one identity for each category.

(A) Those identities which involve one element only on each side of the equality sign:

**Lemma 2.1** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a unipotent groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $(b + c)(x - y) = 0$  for all  $x, y \in \mathbb{Z}_n$ .*

### Proof

$P(x, y)$  satisfies the unipotent law  $\Leftrightarrow P(x, x) = P(y, y) \Leftrightarrow a + bx + cx = a + by + cy \Leftrightarrow a + bx - cx - a - by - cy = 0 \Leftrightarrow (b + c)(x - y) = 0$  as required.

**Lemma 2.2** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a unipotent quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $(b + c)(x - y) = 0$  and  $(b, n) = (c, n) = 1$  for all  $x, y \in \mathbb{Z}_n$ .*

### Proof

This is proved by using Lemma 2.1 and Theorem 1.2

**Theorem 2.2** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a unipotent groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $b + c \equiv 0 \pmod{n}$ .*

### Proof

This is proved by using Lemma 2.1.

**Theorem 2.3** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a unipotent quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $b + c \equiv 0 \pmod{n}$  and  $(b, n) = (c, n) = 1$ .*

### Proof

This is proved by using Lemma 2.2.

**Example 2.1**  $P(x, y) = 5x + y$  is a linear bivariate polynomial over  $\mathbb{Z}_6$ .  $(\mathbb{Z}_6, P)$  is a unipotent groupoid over  $\mathbb{Z}_6$ .

**Example 2.2**  $P(x, y) = 1 + 5x + y$  is a linear bivariate polynomial over  $\mathbb{Z}_6$ .  $(\mathbb{Z}_6, P)$  is a unipotent quasigroup over  $\mathbb{Z}_6$ .

(B) Those identities which involve two elements on one or both sides of the equality sign:

**Lemma 2.3** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a Stein third groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $a(1 + b + c) + x(b^2 + c^2) + y(2bc - 1) = 0$  for all  $x, y \in \mathbb{Z}_n$ .*

**Proof**

$P(x, y)$  satisfies the Stein third law  $\Leftrightarrow P[P(x, y), P(y, x)] = y \Leftrightarrow a(1 + b + c) + x(b^2 + c^2) + y(2bc - 1) = 0$  as required.

**Lemma 2.4** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a Stein third quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $a(1 + b + c) + x(b^2 + c^2) + y(2bc - 1) = 0$  and  $(b, n) = (c, n) = 1$  for all  $x, y \in \mathbb{Z}_n$ .*

**Proof**

This is proved by using Lemma 2.3 and Theorem 1.2

**Theorem 2.4** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a Stein third groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $b^2 + c^2 \equiv 0(\text{mod } n)$ ,  $2bc \equiv 1(\text{mod } n)$  and  $a = 0$ .*

**Proof**

This is proved by using Lemma 2.3.

**Theorem 2.5** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a Stein third quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $b^2 + c^2 \equiv 0(\text{mod } n)$ ,  $2bc \equiv 1(\text{mod } n)$  and  $a = 0$ .*

**Proof**

This is proved by using Lemma 2.4.

**Theorem 2.6** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_p$  such that  $a \neq 0$ .  $P(x, y)$  represents a Stein third groupoid  $(\mathbb{Z}_p, P)$  over  $\mathbb{Z}_p$  if and only if  $b^2 + c^2 \equiv 0(\text{mod } p)$  and  $2bc \equiv 1(\text{mod } p)$ .*

**Proof**

This is proved by using Lemma 2.3.

**Theorem 2.7** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_p$  such that  $a \neq 0$ .  $P(x, y)$  represents a Stein third quasigroup  $(\mathbb{Z}_p, P)$  over  $\mathbb{Z}_p$  if and only if  $b^2 + c^2 \equiv 0(\text{mod } p)$  and  $2bc \equiv 1(\text{mod } p)$ .*

**Proof**

This is proved by using Lemma 2.4.

**Example 2.3**  $P(x, y) = 2x + 3y$  is a linear bivariate polynomial over  $\mathbb{Z}_5$ .  $(\mathbb{Z}_5, P)$  is a Stein third groupoid over  $\mathbb{Z}_5$ .

**Example 2.4**  $P(x, y) = 2x + 3y$  is a linear bivariate polynomial over  $\mathbb{Z}_5$ .  $(\mathbb{Z}_5, P)$  is a Stein third quasigroup over  $\mathbb{Z}_5$ .

(C) Those identities which involve three distinct elements on one or both sides of the equality sign:

**Lemma 2.5** Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents an Abel-Grassman groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $(x - z)(b - c^2) = 0$  for all  $x, z \in \mathbb{Z}_n$ .

**Proof**

$P(x, y)$  satisfies the Abel-Grassman law  $\Leftrightarrow P[x, P(y, z)] = P[z, P(y, x)] \Leftrightarrow P(x, a + by + cz) = P(z, a + by + cx) \Leftrightarrow a + bx + c(a + by + cz) = a + bz + c(a + by + cx) \Leftrightarrow (x - z)(b - c^2) = 0$  as required.

**Lemma 2.6** Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents an Abel-Grassman quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $(x - z)(b - c^2) = 0$  and  $(b, n) = (c, n) = 1$  for all  $x, y, z \in \mathbb{Z}_n$ .

**Proof**

This is proved by using Lemma 2.5 and Theorem 1.2

**Theorem 2.8** Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents an Abel-Grassman groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $c^2 \equiv b \pmod{n}$ .

**Proof**

This is proved by using Lemma 2.5.

**Theorem 2.9** Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents an Abel-Grassman quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $c^2 \equiv b \pmod{n}$  and  $(b, n) = (c, n) = 1$ .

**Proof**

This is proved by using Lemma 2.6.

**Example 2.5**  $P(x, y) = 2 + 4x + 2y$  is a linear bivariate polynomial over  $\mathbb{Z}_6$ .  $(\mathbb{Z}_6, P)$  is an Abel-Grassman groupoid over  $\mathbb{Z}_6$ .

**Example 2.6**  $P(x, y) = 2 + 4x + 2y$  is a linear bivariate polynomial over  $\mathbb{Z}_5$ .  $(\mathbb{Z}_5, P)$  is an Abel-Grassman quasigroup over  $\mathbb{Z}_5$ .

(D) Those involving four elements:

**Lemma 2.7** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents an external medial groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $w(b^2 - c^2) + z(c^2 - b^2) = 0$  for all  $w, z \in \mathbb{Z}_n$ .*

**Proof**

$P(x, y)$  satisfies the external medial law  $\Leftrightarrow P[P(w, x), P(y, z)] = P[P(z, x), P(y, w)]$   
 $\Leftrightarrow a + b(a + bw + cx) + c(a + by + cz) = a + b(a + bz + cx) + c(a + by + cw) \Leftrightarrow$   
 $w(b^2 - c^2) + z(c^2 - b^2) = 0$  as required.

**Lemma 2.8** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents an external medial quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $w(b^2 - c^2) + z(c^2 - b^2) = 0$  and  $(b, n) = (c, n) = 1$  for all  $w, z \in \mathbb{Z}_n$ .*

**Proof**

This is proved by using Lemma 2.7 and Theorem 1.2.

**Theorem 2.10** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $(\mathbb{Z}_n, P)$  represents an external medial groupoid over  $\mathbb{Z}_n$  if and only if  $b^2 \equiv c^2 \pmod{n}$ .*

**Proof**

This is proved by using Lemma 2.7.

**Theorem 2.11** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $(\mathbb{Z}_n, P)$  represents an external medial quasigroup over  $\mathbb{Z}_n$  if and only if  $b^2 \equiv c^2 \pmod{n}$  and  $(b, n) = (c, n) = 1$ .*

**Proof**

This is proved by using Lemma 2.8 and Theorem 1.2.

**Example 2.7**  $P(x, y) = 4 + 2x + 2y$  is a linear bivariate polynomial over  $\mathbb{Z}_6$ .  $(\mathbb{Z}_6, P)$  is an external medial groupoid over  $\mathbb{Z}_6$ .

**Example 2.8**  $P(x, y) = 2 + 8x + y$  is a linear bivariate polynomial over  $\mathbb{Z}_9$ .  $(\mathbb{Z}_9, P)$  is an external medial quasigroup over  $\mathbb{Z}_9$ .

(E) Those involving left or right inverse elements:

**Lemma 2.9** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a cross inverse property groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $a(bc - 1) + x(b^2c + 1 - b - bc) + cy(bc - 1) = 0$  for all  $x, y \in \mathbb{Z}_n$ .*

**Proof**

$P(x, y)$  satisfies the cross inverse property  $\Leftrightarrow P[P(x, y), x^\rho] = y \Leftrightarrow P(a + bx + cy, x^\rho) = y \Leftrightarrow a + b(a + bx + cy) + cx^\rho = y \Leftrightarrow a(bc - 1) + x(b^2c + 1 - b - bc) + cy(bc - 1) = 0$  as required.

**Lemma 2.10** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a cross inverse property quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $a(bc - 1) + x(b^2c + 1 - b - bc) + cy(bc - 1) = 0$  and  $(b, n) = (c, n) = 1$  for all  $x, y, z \in \mathbb{Z}_n$ .*

**Proof**

This is proved by using Lemma 2.9 and Theorem 1.2

**Theorem 2.12** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_p$  such that  $a \neq 0$ .  $P(x, y)$  represents a CIP quasigroup  $(\mathbb{Z}_p, P)$  over  $\mathbb{Z}_p$  if and only if  $bc \equiv 1(\text{mod } p)$ .*

**Proof**

This is proved by using Lemma 2.10.

**Theorem 2.13** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$  such that  $a \neq 0$  and  $c$  is invertible in  $\mathbb{Z}_n$ .  $P(x, y)$  represents a CIP groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $bc \equiv 1(\text{mod } n)$ .*

**Proof**

This is proved by using Lemma 2.9.

**Theorem 2.14** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$  such that  $a \neq 0$ ,  $c$  is invertible in  $\mathbb{Z}_n$  and  $(b, n) = (c, n) = 1$ .  $P(x, y)$  represents a CIP quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if and only if  $bc \equiv 1(\text{mod } n)$ .*

**Proof**

This is proved by using Lemma 2.10.

**Theorem 2.15** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$ .  $P(x, y)$  represents a CIP groupoid  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if  $bc \equiv 1(\text{mod } n)$ .*

**Proof**

This is proved by using Lemma 2.9.

**Theorem 2.16** *Let  $P(x, y) = a + bx + cy$  be a linear bivariate polynomial over  $\mathbb{Z}_n$  such that  $(b, n) = (c, n) = 1$ .  $P(x, y)$  represents a CIP quasigroup  $(\mathbb{Z}_n, P)$  over  $\mathbb{Z}_n$  if  $bc \equiv 1(\text{mod } n)$ .*

**Proof**

This is proved by using Lemma 2.10.

**Example 2.9**  $P(x, y) = 2 + 4x + 4y$  is a linear bivariate polynomial over  $\mathbb{Z}_5$ .  $(\mathbb{Z}_5, P)$  is a cross inverse property groupoid over  $\mathbb{Z}_5$ .

**Example 2.10**  $P(x, y) = 3 + 4x + 4y$  is a linear bivariate polynomial over  $\mathbb{Z}_5$ .  $(\mathbb{Z}_5, P)$  is a cross inverse property quasigroup over  $\mathbb{Z}_5$ .

S/N	NAME	G	Q	$\mathbb{Z}_n$	$\mathbb{Z}_p$	HYPO	N AND S	EXAMPLE
1	Idempotent	✓		✓			$b + c = 1, a = 0$	$5x + 2y, \mathbb{Z}_6$
2	Unipotent	✓		✓			$b + c = 0$	$2 + 4x + 2y, \mathbb{Z}_6$
			✓	✓			$b + c = 0, (b, n) = (c, n) = 1$	$2 + 5x + y, \mathbb{Z}_6$
3	Commut	✓		✓			$b = c$	$1 + 4x + 4y, \mathbb{Z}_6$
			✓	✓			$b = c, (b, n) = (c, n) = 1$	$1 + 5x + 5y, \mathbb{Z}_6$
4	Sade Right	✓			✓	$a \neq 0$	$b = -1$	$2 + 6x + 4y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = -1$	$1 + 5x + 4y, \mathbb{Z}_7$
5	Sade Left	✓			✓	$a \neq 0$	$c = -1$	$2 + 4x + 5y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$c = -1$	$2 + 5x + 5y, \mathbb{Z}_7$
6	Right Alternative	✓			✓	$a \neq 0$	$b = c = 1$	$3 + x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = c = 1$	$3 + x + y, \mathbb{Z}_7$
7	Left Alternative	✓			✓	$a \neq 0$	$b = c = 1$	$2 + x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = c = 1$	$2 + x + y, \mathbb{Z}_7$
8	Medial Alternative	✓			✓	$a \neq 0$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_7$
		✓			✓	$b \neq c$	$b + c = 1$	$2 + 4x + 2y, \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_7$
			✓		✓	$b \neq c$	$b + c = 1$	$2 + 4x + 2y, \mathbb{Z}_7$
9	Right Semi Symmetry	✓			✓	$a \neq 0$	$b = c = -1$	$2 + 4x + 4y, \mathbb{Z}_5$
		✓		✓		$a = 0$	$bc = 1, c^2 = -b$	$5x + 2y, \mathbb{Z}_9$
			✓		✓	$a \neq 0$	$b = c = -1$	$2 + 4x + 4y, \mathbb{Z}_5$
			✓	✓		$a = 0$	$bc = 1, c^2 = -b$	$5x + 2y, \mathbb{Z}_9$
10	Left Semi Symmetry	✓			✓	$a \neq 0$	$b = c = -1$	$3 + 4x + 4y, \mathbb{Z}_5$
		✓		✓		$a = 0$	$b = 1, b^2 = -c$	$x + 9y, \mathbb{Z}_{10}$
			✓		✓	$a \neq 0$	$b = c = -1$	$3 + 4x + 4y, \mathbb{Z}_5$
			✓	✓		$a = 0$	$b = 1, b^2 = -c$	$x + 9y, \mathbb{Z}_{10}$
11	Stein First	✓			✓	$a \neq 0$	$b = c$	$3 + 4x + 4y, \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_5$
12	Stein Second	✓			✓	$a \neq 0$	$b = c$	$3 + 4x + 4y, \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_5$
13	Schroder Second	✓		✓			$b^2 + c^2 = 1, 2bc = 0, a = 0$	$2x + 3y, \mathbb{Z}_6$
			✓	✓			$b^2 + c^2 = 1, 2bc = 0, a = 0, (b, n) = (c, n) = 1$	?
		✓			✓	$a \neq 0$	$b + c = -1, b^2 + c^2 = 1, 2bc = 0$	?
			✓		✓	$a \neq 0$	$b + c = -1, b^2 + c^2 = 1, 2bc = 0$	?



14	Stein Third	✓		✓			$b^2 + c^2 = 0, 2bc = 1, a = 0$	?
			✓	✓			$(b, n) = (c, n) = 1, b^2 + c^2 = 0, 2bc = 1, a = 0$	?
		✓			✓	$a \neq 0$	$b^2 + c^2 = 0, 2bc = 1,$	$3 + 2x + 4y, \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b^2 + c^2 = 0, 2bc = 1,$	$2 + 2x + 4y, \mathbb{Z}_5$
15	Associative	✓			✓	$a \neq 0$	$b = c = 1$	$2 + x + y, \mathbb{Z}_6$
			✓		✓	$a \neq 0$	$b = c = 1$	$2 + x + y, \mathbb{Z}_6$
16	Slim	✓		✓		$a = 0, c \text{ invert}$	$bc = 0, c = 1$	!
			✓	✓		$a = 0, c \text{ invert}$	$bc = 0, c = 1, (b, n) = (c, n) = 1$	?
17	Cyclic Associativity	✓		✓			$b = c = 1$	$3 + x + y, \mathbb{Z}_6$
			✓	✓			$b = c = 1, (b, n) = (c, n) = 1$	$3 + x + y, \mathbb{Z}_6$
18	Right Permutability	✓		✓			$b = 1$	$1 + x + 5y, \mathbb{Z}_6$
			✓	✓			$b = 1, (b, n) = (c, n) = 1$	$1 + x + 5y, \mathbb{Z}_6$
19	Left Permutability	✓		✓			$c = 1$	$1 + 5x + y, \mathbb{Z}_6$
			✓	✓			$c = 1, (b, n) = (c, n) = 1$	$3 + 5x + y, \mathbb{Z}_6$
20	Abel Grassman	✓		✓			$c^2 = b$	$2 + 4x + 2y, \mathbb{Z}_6$
			✓	✓			$c^2 = b, (b, n) = (c, n) = 1$	$2 + 4x + 2y, \mathbb{Z}_9$
21	Commuting Product	✓			✓	$a \neq 0$	$b = c = 1$	$1 + x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = c = 1$	$1 + x + y, \mathbb{Z}_7$
22	Dual Comm Product	✓			✓	$a \neq 0$	$b = c = 1$	$1 + x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = c = 1$	$1 + x + y, \mathbb{Z}_7$
23	Right Transitivity	✓			✓	$a \neq 0$	$b = 1, c = -1$	$2 + x + 6y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = 1, c = -1$	$2 + x + 6y, \mathbb{Z}_7$
24	Left Transitivity	✓			✓	$a \neq 0$	$b = -1, c = 1$	$2 + 6x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = -1, c = 1$	$2 + 6x + y, \mathbb{Z}_7$
25	Schweitzer	✓		✓		$b, c \text{ invert}$	$b = 1, c = -1$	$2 + x + 5y, \mathbb{Z}_6$
			✓	✓		$b, c \text{ invert}$	$b = 1, c = -1, (b, n) = (c, n) = 1$	$2 + x + 5y, \mathbb{Z}_6$
		✓			✓	$a \neq 0$	$b = 1, c = -1$	$3 + x + 6y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = 1, c = -1$	$3 + x + 6y, \mathbb{Z}_7$
26	Dual of Schweitzer	✓		✓		$b, c \text{ invert}$	$b = 1, c = -1$	$2 + x + 5y, \mathbb{Z}_6$
			✓	✓		$b, c \text{ invert}$	$b = 1, c = -1, (b, n) = (c, n) = 1$	$2 + x + 5y, \mathbb{Z}_6$
		✓			✓	$a \neq 0$	$b = 1, c = -1$	$3 + x + 6y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = 1, c = -1$	$3 + x + 6y, \mathbb{Z}_7$

27	Right Self Distributive	✓			✓		$c = 1 - b, a = 0$	$3x + 5y, \mathbb{Z}_7$
			✓		✓		$c = 1 - b, a = 0$	$3x + 5y, \mathbb{Z}_7$
28	Left Self Distributive	✓			✓		$c = 1 - b, a = 0$	$3x + 5y, \mathbb{Z}_7$
			✓		✓		$c = 1 - b, a = 0$	$3x + 5y, \mathbb{Z}_7$
29	Right Abelian Distributivity	✓		✓		$b, c$ invert	$b = c, 2b^2 = b$	?
			✓	✓		$b, c$ invert	$b = c, 2b^2 = b$	?
		✓		✓		$a \neq 0$	$b = c, 2b^2 = b$	?
			✓	✓		$a \neq 0$	$b = c, 2b^2 = b$	?
		✓			✓	$a \neq 0$	$b = c, 2b = 1$	$2 + 3x + 3y, \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c, 2b = 1$	$2 + 3x + 3y, \mathbb{Z}_5$
30	Left Abelian Distributivity	✓		✓		$b, c$ invert	$b = c, 2b^2 = b$	
			✓	✓		$b, c$ invert	$b = c, 2b^2 = b$	?
		✓		✓		$a \neq 0$	$b = c, 2b^2 = b$	?
			✓	✓		$a \neq 0$	$b = c, 2b^2 = b$	?
		✓			✓	$a \neq 0$	$b = c, 2b = 1$	$2 + 3x + 3y, \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c, 2b = 1$	$2 + 3x + 3y, \mathbb{Z}_5$
31	Bol Moufang	✓		✓			$b = c = 1$	$2 + x + y, \mathbb{Z}_6$
			✓	✓		$(b, n) = (c, n) = 1$	$b = c = 1$	$2 + x + y, \mathbb{Z}_6$
32	Dual Bol Moufang	✓		✓			$b = c = 1$	$2 + x + y, \mathbb{Z}_6$
			✓	✓		$(b, n) = (c, n) = 1$	$b = c = 1$	$2 + x + y, \mathbb{Z}_6$
33	Moufang	✓			✓		$b = c = 1, a = 0$	$x + y, \mathbb{Z}_5$
			✓		✓		$b = c = 1, a = 0$	$x + y, \mathbb{Z}_5$
34	R Bol	✓			✓	$a \neq 0$	$b^2 = 1, b = c = 1$	$2 + x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b^2 = 1, b = c = 1$	$2 + x + y, \mathbb{Z}_7$
		✓			✓	$-1 \neq b \neq c$	$b^2 = 1, c = 1, a = 0$	$8x + y, \mathbb{Z}_{63}$
			✓		✓	$-1 \neq b \neq c$	$b^2 = 1, c = 1, a = 0$	$8x + y, \mathbb{Z}_{63}$
35	L Bol	✓			✓	$a \neq 0$	$c^2 = 1, b = c = 1$	$2 + x + y, \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$c^2 = 1, b = c = 1$	$2 + x + y, \mathbb{Z}_7$
		✓			✓	$-1 \neq b \neq c$	$c^2 = 1, b = 1, a = 0$	$x + 8y, \mathbb{Z}_{63}$
			✓		✓	$-1 \neq b \neq c$	$c^2 = 1, b = 1, a = 0$	$x + 8y, \mathbb{Z}_{63}$

36	RC <sub>4</sub>	✓			✓	$a = 0$	$c = b^2 = 1$	$8x + y, \mathbb{Z}_{63}$
			✓		✓	$a = 0$	$c = b^2 = 1$	$8x + y, \mathbb{Z}_{63}$
		✓		✓		$a = 0, b, c \text{ invert}$	$c = b^2 = 1$	$8x + y, \mathbb{Z}_{63}$
			✓	✓		$a = 0, b, c \text{ invert}$	$c = b^2 = 1, (b, n) = (c, n) = 1$	$8x + y, \mathbb{Z}_{63}$
		✓		✓			$b = -1, c = 1$	$2 + 5x + y, \mathbb{Z}_6$
			✓	✓			$b = -1, c = 1, (b, n) = (c, n) = 1$	$2 + 5x + y, \mathbb{Z}_6$
37	LC <sub>4</sub>	✓			✓	$a = 0$	$b = c^2 = 1$	$x + 8y, \mathbb{Z}_{63}$
			✓		✓	$a = 0$	$b = c^2 = 1$	$x + 8y, \mathbb{Z}_{63}$
		✓		✓		$a = 0, b, c \text{ invert}$	$b = c^2 = 1$	$x + 3y, \mathbb{Z}_8$
			✓	✓		$a = 0, b, c \text{ invert}$	$b = c^2 = 1, (b, n) = (c, n) = 1$	$x + 4y, \mathbb{Z}_{15}$
		✓		✓			$b = -1, c = 1$	$2 + 5x + y, \mathbb{Z}_6$
			✓	✓			$b = -1, c = 1, (b, n) = (c, n) = 1$	$2 + 5x + y, \mathbb{Z}_6$
38	RC <sub>1</sub>	✓			✓	$a = 0$	$c = b^2 = 1$	$8x + y, \mathbb{Z}_{63}$
			✓		✓	$a = 0$	$c = b^2 = 1$	$8x + y, \mathbb{Z}_{63}$
		✓		✓		$a = 0, b, c \text{ invert}$	$c = b^2 = 1$	$8x + y, \mathbb{Z}_{63}$
			✓	✓		$a = 0, b, c \text{ invert}$	$c = b^2 = 1, (b, n) = (c, n) = 1$	$8x + y, \mathbb{Z}_{63}$
		✓		✓			$b = -1, c = 1$	$2 + 5x + y, \mathbb{Z}_6$
			✓	✓			$b = -1, c = 1, (b, n) = (c, n) = 1$	$2 + 5x + y, \mathbb{Z}_6$
39	LC <sub>1</sub>	✓			✓	$a = 0, c \neq 1$	$c = -1$	$3x + 6y, \mathbb{Z}_7$
			✓		✓	$a = 0, c \neq 1$	$c = -1$	$3x + 6y, \mathbb{Z}_7$
		✓		✓		$a = 0, c \neq 1, c \text{ invert}$	$c = -1$	$5x + 5y, \mathbb{Z}_6$
			✓	✓		$a = 0, c \neq 1, c \text{ invert}$	$c = -1, (b, n) = (c, n) = 1$	$5x + 5y, \mathbb{Z}_6$
40	LC <sub>3</sub>	✓		✓			$c = 1, b = -2$	$3 + 4x + y, \mathbb{Z}_6$
			✓	✓			$c = 1, b = -2, (b, n) = (c, n) = 1$	$2 + 5x + y, \mathbb{Z}_7$
41	RC <sub>3</sub>	✓		✓			$c = 1, b = -2$	$3 + 4x + y, \mathbb{Z}_6$
			✓	✓			$c = 1, b = -2, (b, n) = (c, n) = 1$	$2 + 5x + y, \mathbb{Z}_7$
42	C-Law	✓			✓	$a = 0$	$b = c = -1$	$4x + 4y, \mathbb{Z}_5$
			✓		✓	$a = 0$	$b = c = -1$	$4x + 4y, \mathbb{Z}_5$
		✓		✓		$a \neq 0, b \neq 1, b, c \text{ inv}$	$b = c = -1$	$3 + 5x + 5y, \mathbb{Z}_6$
			✓	✓		$a \neq 0, b \neq 1, b, c \text{ inv}$	$b = c = -1, (b, n) = (c, n) = 1$	$3 + 5x + 5y, \mathbb{Z}_6$
43	LIP	✓			✓	$a \neq 0$	$c^2 = b^2 = bc = 1$	?
			✓		✓	$a \neq 0$	$c^2 = b^2 = bc = 1$	?
44	RIP	✓			✓	$a \neq 0$	$c^2 = b^2 = bc = 1$	?
			✓		✓	$a \neq 0$	$c^2 = b^2 = bc = 1$	?

45	1st Right CIP	✓			✓	$a \neq 0$	$bc = 1$	$2 + 3x + 4y, \mathbb{Z}_{11}$
			✓		✓	$a \neq 0$	$bc = 1$	$2 + 3x + 4y, \mathbb{Z}_{11}$
		✓		✓		$a \neq 0, c \text{ inv}$	$bc = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
			✓	✓		$a \neq 0, c \text{ inv}$	$bc = 1, (b, n) = (c, n) = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
46	2nd Right CIP	✓		✓			$bc = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
			✓	✓			$bc = 1, (b, n) = (c, n) = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
47	1st Left CIP	✓			✓	$a \neq 0$	$bc = 1$	$2 + 3x + 4y, \mathbb{Z}_{11}$
			✓		✓	$a \neq 0$	$bc = 1$	$2 + 3x + 4y, \mathbb{Z}_{11}$
		✓		✓		$a \neq 0, b \text{ inv}$	$bc = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
			✓	✓		$a \neq 0, b \text{ inv}$	$bc = 1, (b, n) = (c, n) = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
48	2nd Left CIP	✓		✓			$bc = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
			✓	✓			$bc = 1, (b, n) = (c, n) = 1$	$3 + 3x + 3y, \mathbb{Z}_8$
49	R AAIP	✓			✓	$bc + b \neq 1$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_{11}$
			✓		✓	$bc + b \neq 1$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_{11}$
		✓			✓	$c \neq b$	$b + bc = 1$	$2 + 3x + y, \mathbb{Z}_5$
			✓		✓	$c \neq b$	$b + bc = 1$	$2 + 3x + y, \mathbb{Z}_5$
50	L AAIP	✓			✓	$bc + b \neq 1$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_{11}$
			✓		✓	$bc + b \neq 1$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_{11}$
		✓			✓	$c \neq b$	$b + bc = 1$	$2 + 3x + y, \mathbb{Z}_5$
			✓		✓	$c \neq b$	$b + bc = 1$	$2 + 3x + y, \mathbb{Z}_5$
51	R AIP	✓		✓				$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
52	L AIP	✓		✓				$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
53	R SAIP	✓		✓				$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$

54	L SAIP	✓		✓				$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
55	R WIP	✓			✓	$a = 0, c^2 \neq 0$	$bc = 1$	$3x + 5y, \mathbb{Z}_7$
			✓		✓	$a = 0, c^2 \neq 0$	$bc = 1$	$3x + 5y, \mathbb{Z}_7$
		✓		✓		$a = 0, c \text{ inv}$	$bc = 1$	$3x + 4y, \mathbb{Z}_6$
			✓	✓		$a = 0, c \text{ inv}$	$bc = 1, (b, n) = (c, n) = 1$	?
		✓		✓		$a = 0, bc + b \neq 1$	$bc = 1$	?
			✓	✓		$a = 0, bc + b \neq 1$	$bc = 1, (b, n) = (c, n) = 1$	?
56	L WIP	✓			✓	$a = 0, b^2 \neq 0$	$bc = 1$	$3x + 5y, \mathbb{Z}_7$
			✓		✓	$a = 0, b^2 \neq 0$	$bc = 1$	$3x + 5y, \mathbb{Z}_7$
		✓		✓		$a = 0, b \text{ inv}$	$bc = 1$	$3x + 4y, \mathbb{Z}_6$
			✓	✓		$a = 0, b \text{ inv}$	$bc = 1, (b, n) = (c, n) = 1$	?
		✓		✓		$a = 0, bc + c \neq 1$	$bc = 1$	?
			✓	✓		$a = 0, bc + c \neq 1$	$bc = 1, (b, n) = (c, n) = 1$	?
57	$E_l$	✓	✓					$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
58	$E_r$	✓	✓					$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
59	Right F	✓	✓					$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
60	Left F	✓	✓					$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$

61	Medial	✓	✓					$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
62	Specialized Medial	✓	✓					$a + bx + cy, \mathbb{Z}_n$
			✓	✓				$a + bx + cy, \mathbb{Z}_n$
		✓			✓			$a + bx + cy, \mathbb{Z}_n$
			✓		✓			$a + bx + cy, \mathbb{Z}_n$
63	First Rectangle	✓			✓		$b = c$	$2 + 4x + 4y, \mathbb{Z}_7$
			✓		✓		$b = c$	$2 + 4x + 4y, \mathbb{Z}_7$
		✓		✓		$c \text{ inv}$	$b = c$	$2 + 4x + 4y, \mathbb{Z}_6$
			✓	✓		$c \text{ inv}$	$b = c, (b, n) = (c, n) = 1$	$2 + 4x + 4y, \mathbb{Z}_6$
64	Second Rectangle	✓			✓		$b = -c$	$2 + 4x + 4y, \mathbb{Z}_7$
			✓		✓		$b = -c$	$2 + 4x + 4y, \mathbb{Z}_7$
		✓		✓		$b \text{ inv}$	$b = -c$	$2 + 4x + 4y, \mathbb{Z}_6$
			✓	✓		$b \text{ inv}$	$b = -c, (b, n) = (c, n) = 1$	$2 + 4x + 4y, \mathbb{Z}_6$
65	$C_i, i = 1 - 6$		✓		✓		$b = c$	$3 + 5x + 5y, \mathbb{Z}_7$
66	$CM_i, i = 1 - 14$		✓		✓	$b \neq -c$	$b = c$	$3 + 5x + 5y, \mathbb{Z}_7$

Table 1: A Table for the Characterization of Varieties of Groupoids and Quasigroups Generated by  $P(x, y)$  over  $\mathbb{Z}_n$

**Remark 2.1** A summary of the results on the characterization of groupoids and quasigroups generated by  $P(x, y)$  is exhibited in Table 1. In the table "G" stands for "groupoid", "Q" stands for "quasigroup", "HYPO" stands for "hypothesis", "N AND S" stands for "necessary and sufficient condition(s)". Cells with question marks mean examples could not be gotten.

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